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ON AN INVERSE PROBLEM OF BENDING OF A PHYSICALLY NONLINEAR INHOMOGENEOUS PLATE

I. Yu. Tsvelodub

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An elastic plate with a physically nonlinear inclusion of an arbitrary shape is considered. This plate is subjected to pure bending under the action of transverse forces and bending moments applied at the external boundary of the plate. There are no loads distributed over the surface. The problem of finding external actions that provide a necessary uniform moment state in the inclusion, i.e., prescribed constant moments and curvatures, is formulated and solved.

Key words: *pure bending of an inhomogeneous plate, physically nonlinear inclusion of an arbitrary form, uniform field of moments.*

An inverse problem for an elastic region with a physically nonlinear inclusion (PNI) under conditions of plane deformation or in a generalized plane stress state was considered in [1]. The task was to obtain a homogeneous stress–strain state in the PNI by choosing appropriate external forces. The problem is extended in the present paper to the case of pure bending of an elastic plate with an arbitrarily shaped PNI. The solution is constructed in a closed form.

1. Formulation of the Problem. We consider a plate of constant thickness h whose mid-surface is the region $S = S_1 \cup S_2$ in the plane Ox_1x_2; S_1 and S_2 are a physically nonlinear singly connected region and an isotropic linearly elastic doubly connected region with external boundaries L_1 and L_2 (L_1 separates the regions S_2 and S_1). External actions (transverse forces and bending moments) are applied only at the boundary L_2, and there are no surface loads. Thus, the plate is under conditions of pure bending, and its strains ε_{kl} are described by the known relations [2]

$$\begin{gathered} \varepsilon_{kl}(x_1, x_2, x_3) = -x_3 w_{,kl}, \qquad k, l = 1, 2, \\ w = w(x_1, x_2), \qquad (x_1, x_2) \in S, \quad |x_3| \le h/2 \end{gathered} \tag{1.1}$$

(the subscripts after the comma indicate the derivatives with respect to the corresponding coordinate). The equilibrium equations take the form

$$\begin{gathered} Q_k = M_{kl,l}, \qquad Q_{k,k} = 0, \\ Q_k = \int_{-h/2}^{h/2} \sigma_{3k}\, dx_3, \qquad M_{kl} = \int_{-h/2}^{h/2} \sigma_{kl} x_3\, dx_3, \qquad k, l = 1, 2. \end{gathered} \tag{1.2}$$

In formulas (1.1) and (1.2), w is the deflection, Q_k and M_{kl} are the shear forces and moments, respectively, and σ_{kl} are the stress components; summation is performed over repeated indices from 1 to 2. The coordinate system Ox_1x_2 is chosen so that $(0, 0) \in S_1$.

The conditions of continuity of the deflection w, angles of mid-surface turning (i.e., normal derivatives $\partial w/\partial n$), bending moments G, and transverse forces $Q + \partial H/\partial s$ are satisfied at the boundary L_1 between the

Lavrent'ev Institute of Hydrodynamics, Siberian Division, Russian Academy of Sciences, Novosibirsk 630090; itsvel@hydro.nsc.ru. Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, Vol. 48, No. 5, pp. 104–107, September–October, 2007. Original article submitted October 6, 2006.

regions S_1 and S_2. Here $G = M_{kl} n_k n_l$, $Q = Q_k n_k$, $H = M_{kl} n_k t_l$ is the torque, n_k and t_k are the components of the unit vectors of the normal and the tangent to the contour L_1, respectively, and s is the length of the contour arc [3].

The following equalities are valid in the elastic region S_2 [3]:

$$M_{kl} = -D[(1-\nu)w_{,kl} + \nu w_{,nn}\delta_{kl}], \qquad Q_k = -Dw_{,nnk}, \qquad k,l = 1,2,$$
$$D = Eh^3/[12(1-\nu^2)]. \tag{1.3}$$

Here δ_{kl} are the components of the plane unit tensor, D is the cylindrical rigidity, E is Young's modulus, and ν is Poisson's ratio.

The constitutive equations for the physically nonlinear inclusion S_1 have the form [2]

$$\sigma_{kl} = F_{kl}(\varepsilon_{mn}), \qquad \varepsilon_{kl} = x_3 \varkappa_{kl}, \qquad \varkappa_{kl} = -w_{,kl},$$
$$F_{kl}(-\varepsilon_{mn}) = -F_{kl}(\varepsilon_{mn}), \qquad k,l,m,n = 1,2, \tag{1.4}$$

where F_{kl} are the nonlinear differentiable functions satisfying (at $\xi_{kl}\xi_{kl} \neq 0$) the inequality $\partial F_{kl}/\partial\varepsilon_{mn}\xi_{kl}\xi_{mn} > 0$, which is equivalent to the condition of stability of the PNI deformation process [1, 2]:

$$\Delta\sigma_{kl}\Delta\varepsilon_{kl} > 0 \qquad \text{for} \quad \Delta\varepsilon_{kl}\Delta\varepsilon_{kl} \neq 0. \tag{1.5}$$

As was demonstrated in [2], inequality (1.5) ensures unequivocal solubility of the relations

$$M_{kl} = 2\int\limits_0^{h/2} F_{kl}(x_3\varkappa_{mn})x_3\,dx_3$$

following from (1.2) and (1.4), with respect to the curvatures $\varkappa_{kl}$.

In addition, the uniformity of the field of moments M_{kl} in S_1 ensures the uniformity of the field of curvatures $\varkappa_{kl}$ in S_1 [2].

An example of functions (1.4) satisfying condition (1.5) can be found in [2].

Let us formulate the main problem: Which transverse forces and bending moments have to be applied to the external boundary L_2 of the region S_2 to reach a necessary uniform moment state in the PNI, i.e., for M_{kl} in S_1 to be independent of x_k $(k,l = 1,2)$?

2. Solution of the Problem. It follows from Eqs. (1.2) and (1.3) that the deflection w in the region S_2 in the case considered with no surface loads satisfies the biharmonic equation $w_{,kkll} = 0$; therefore, the following known presentation holds [4]:

$$2w = \bar{z}\varphi(z) + z\overline{\varphi(z)} + \chi(z) + \overline{\chi(z)}, \qquad z = x_1 + ix_2. \tag{2.1}$$

According to (2.1), the deflection w can be considered as a function of two independent complex variables z and $\bar{z}$. Then, we obtain the following relations for the moments M_{kl} in S_2 [3–5]:

$$M_{11} + M_{22} = -2D(1+\nu)[\Phi(z) + \overline{\Phi(z)}\,] = -4D(1+\nu)w_{,z\bar{z}},$$
$$M_{22} - M_{11} + 2iM_{12} = 2D(1-\nu)[\bar{z}\,\Phi\prime(z) + \Psi(z)] = 4D(1-\nu)w_{,zz}, \tag{2.2}$$
$$\Phi(z) = \varphi'(z), \qquad \Psi(z) = \chi''(z).$$

In the region S_1, we have $w_{,kl} = -\varkappa_{kl}$, and $w_{,kl}$ are independent of x_1 and x_2. With accuracy to a function linear with respect to x_1 and x_2, we can find

$$2w = -\varkappa_{kl}x_k x_l.$$

Passing to the variables $z = x_1 + ix_2$ and $\bar{z} = x_1 - ix_2$, we obtain

$$8w(z,\bar{z}) = (\varkappa_{22} - \varkappa_{11} + 2i\varkappa_{12})z^2 + (\varkappa_{22} - \varkappa_{11} - 2i\varkappa_{12})\bar{z}^2 - 2(\varkappa_{11} + \varkappa_{22})z\bar{z}. \tag{2.3}$$

The boundary conditions on L_1 for the functions $\Phi(z)$ and $\Psi(z)$ from (2.2), which determine the stress–stain state in S_2, have the form [2, 3]

$$\Phi(\tau) - \lambda_k\overline{\Phi(\tau)} - [\bar{\tau}\Phi'(\tau) + \Psi(\tau)]\,\mathrm{e}^{2i\alpha} = f_k, \qquad k = 1,2,$$

$$\lambda_1 = -1, \qquad \lambda_2 = \frac{3+\nu}{1-\nu}, \qquad f_1 = \mathrm{e}^{i\alpha}\,\frac{d(w_{,2}+iw_{,1})}{ds} = 2i\,\mathrm{e}^{i\alpha}\,\frac{dw_{,z}}{ds}, \tag{2.4}$$

$$f_2 = \frac{1}{D(1-\nu)}\Big[G - i\Big(H + \int\limits_0^s Q\,ds\Big)\Big],$$

where α is the angle between the normal to the contour L_1 at the point τ and the axis Ox_1. The quantities $w_{,k}$ ($k = 1,2$), $w_{,z}$, G, H, and Q in (2.4) are assumed to be defined on L_1 as functions of the arc coordinate s. By virtue of the above-indicated conditions of continuity, these functions can be determined by approaching L_1 from the region S_1, where the stress–strain state is prescribed, i.e., M_{kl} and $\varkappa_{kl}$ are known, and the deflection w is found from (2.3).

As was shown in [2], the following equalities are valid for the quantities f_1 and f_2 from (2.4):

$$\begin{gathered} f_1 = -[\varkappa_{11} + \varkappa_{22} + (\varkappa_{22} - \varkappa_{11} + 2i\varkappa_{12})\,\mathrm{e}^{2i\alpha}]/2, \\ f_2 = [M_{11} + M_{22} - (M_{22} - M_{11} + 2iM_{12})\,\mathrm{e}^{2i\alpha}]/[2D(1-\nu)]. \end{gathered} \tag{2.5}$$

Let the conformal mapping of an infinite region located outside the boundary L_1 and including the region S_2 onto the exterior of the unit circumference γ_1 of the complex plane ζ have the form

$$z = \omega(\zeta) = m_1\zeta + \sum_{k=1}^{\infty} m_{-k}\zeta^{-k}, \qquad \zeta = \rho\,\mathrm{e}^{i\theta}\,. \tag{2.6}$$

Then, from Eqs. (2.4)–(2.6) with $\rho = 1$, we obtain the boundary conditions for the functions $\Phi_1(\zeta) = \Phi(\omega(\zeta))$ and $\Psi_1(\zeta) = \Psi(\omega(\zeta))$ determining the stress–strain state for $|\zeta| > 1$ [4, 5]:

$$\begin{gathered} \Phi_1(\sigma) - \lambda_k\overline{\Phi_1(\sigma)} - [\overline{\omega(\sigma)}\,\Phi_1'(\sigma)/\omega'(\sigma) + \Psi_1(\sigma)]\,\mathrm{e}^{2i\alpha} = \overline{F_k(\sigma)} \quad \text{on} \quad \gamma_1, \\ \overline{F_k(\sigma)} = f_k \quad (k = 1,2), \qquad \sigma = \mathrm{e}^{i\theta}, \qquad \mathrm{e}^{2i\alpha} = \sigma^2\omega'(\sigma)/\overline{\omega'(\sigma)}. \end{gathered} \tag{2.7}$$

It follows from (2.7) that

$$\Phi_1(\sigma) = (1-\nu)[F_1(\sigma) - F_2(\sigma)]/4 \qquad \text{on} \quad \gamma_1.$$

From here, with allowance for (2.5), we find [4, 5]

$$\begin{gathered} \Phi_1(\zeta) = A - (B + iC)\omega_1'(\zeta)/\omega'(\zeta), \qquad \omega_1(\zeta) \equiv \bar{\omega}(\zeta^{-1}) = \overline{\omega(\bar{\zeta}^{-1})}, \\ \Psi_1(\zeta) = \{[\bar{F}_1(\zeta^{-1}) - \Phi_1(\zeta) - \bar{\Phi}_1(\zeta^{-1})]\omega_1'(\zeta) - \Phi_1'(\zeta)\omega_1(\zeta)\}/\omega'(\zeta), \\ \bar{F}_1(\zeta^{-1}) = A_1 - (B_1 - iC_1)\omega'(\zeta)/\omega_1'(\zeta), \qquad |\zeta| > 1, \\ 8A = 2(1-\nu)A_1 - (M_{11} + M_{22})/D, \qquad 8B = 2(1-\nu)B_1 + (M_{22} - M_{11})/D, \\ 4C = (1-\nu)C_1 - M_{12}/D, \quad 2A_1 = -(\varkappa_{11} + \varkappa_{22}), \quad 2B_1 = -(\varkappa_{22} - \varkappa_{11}), \quad C_1 = \varkappa_{12}. \end{gathered} \tag{2.8}$$

It follows from (2.6) and (2.8) that the functions $\Phi_1(\zeta)$ and $\Psi_1(\zeta)$ are holomorphic in the ring $1 < |\zeta| < R$, where $R^{-1} = \lim |m_{-n}|^{1/n}$ as $n \to \infty$. If the number of terms under the summation sign in (2.6) is finite, then $R = \infty$. If the contour γ_2 of the plane ζ corresponding to the boundary L_2 lies inside this ring, the sought external actions on L_2 (transverse forces and bending moments) are determined by the known formulas of the form (2.4) [3, 4].

It should be noted that the solution constructed in the region S_2 can be continued beyond the boundary L_2, like in [1], if the corresponding values are $|\zeta| < R$. In particular, if the contour L_1 is an ellipse, the continuation is also possible for $|\zeta| \to \infty$. This corresponds to the case of pure bending of the plate with an elliptic PNI under the action of uniformly distributed moments at infinity [2].

3. Uniqueness of the Problem Solution. It follows from dependences (2.8) that the solution for the stress–strain state in S_2 exists and is unique if the stress–strain state in S_1 is prescribed and the contour γ_2 corresponding to L_2 lies inside the ring $1 < |\zeta| < R$ [1]. The reverse statement is also valid: for prescribed bending moments G and transverse forces $Q + \partial H/\partial s$ on L_2, the moments M_{kl} and curvatures $\varkappa_{kl}$ in $S = S_1 \cup S_2$ are determined uniquely, i.e., the above-considered stress–strain state is formed in the region S under the action of the force actions on L_2 found; hence, the PNI is in a uniform moment state.

The proof of this statement is similar to that given in [1]. In this case, we use condition (1.5) and the equation of virtual works, which has the following form in the case considered with pure bending of plates under the above-indicated conditions of continuity on L_1 [2]:

$$\int\limits_{-h/2}^{h/2} \int\limits_{S} \sigma_{kl}\varepsilon_{kl}\, dS\, dx_3 = \int\limits_{L_2} \Big[\Big(Q + \frac{\partial H}{\partial s}\Big) w - G\, \frac{\partial w}{\partial n}\Big] ds.$$

This equation is valid for all fields of σ_{kl} and ε_{kl} independent of each other; thereby, ε_{kl} and w satisfy relations (1.1), while Q_k and M_{kl} satisfy the equilibrium equations (1.2). The same equation was used in [2] to prove the uniqueness of the solution of the problem on pure bending of an infinite plate with an elliptic PNI.

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